A WEAK NOTION OF STRICT PSEUDO-CONVEXITY. APPLICATIONS AND EXAMPLES.

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Abstract. Let Ω be a bounded $\mathcal{C}^\infty$-smoothly bounded domain in $\mathbb{C}^n$. For such a domain we define a new notion between strict pseudo-convexity and pseudo-convexity: the size of the set W of weakly pseudo-convex points on $\partial\Omega$ is small with respect to Minkowski dimension: near each point in the boundary $\partial\Omega$, there is at least one complex tangent direction in which the slices of W has a upper Minkowski dimension strictly smaller than 2. We propose to call this notion *strong pseudo-convexity*; this word is free since "*strict pseudo-convexity*" gets the precedence in the case where all the points in $\partial\Omega$ are stricly pseudo-convex.

For such domains we prove that if S is a separated sequence of points contained in the support of a divisor in the Blaschke class, then a canonical measure associated to S is bounded. If moreover the domain is p-regular, and the sequence S is dual bounded in the Hardy space $H^p(\Omega)$, then the previous measure is Carleson.

As an application we prove a theorem on interpolating sequences in bounded convex domains of finite type in $\mathbb{C}^n$.

Examples of such pseudo-convex domains are finite type domains in $\mathbb{C}^2$, finite type convex domains in $\mathbb{C}^n$, finite type domains which have locally diagonalizable Lévi form, domains with real analytic boundary and of course, stricly pseudo-convex domains in $\mathbb{C}^n$.

Domains like $|z_1|^2+\exp\{1-|z_2|^{-2}\}<1$, which are not of finite type are nevertheless strongly pseudo-convex, in this sense.

Contents

1. Introduction.

Let $H^\infty(\mathbb{D})$ be the algebra of bounded holomorphic functions in the unit disc $\mathbb{D}$ in $\mathbb{C}$. An important property, which is the key in the characterization by L. Carleson of the interpolating sequences S for $H^\infty(\mathbb{D})$, is the fact that if S is dual bounded for $H^\infty(\mathbb{D})$, (the definition will be given later) then a measure μ_S, canonically associated to the sequence S, is a Carleson measure.

A related result was obtained by N. Varopoulos [28]: he proved that if S is interpolating (which implies that S is dual bounded) in some spaces of bounded functions, then the measure μ_S is Carleson and P. Thomas [26] proved that if S is interpolating for $H^p(\mathbb{B})$, the Hardy space of holomorphic functions in the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, then the associated measure μ_S is still Carleson. I gave a proof of the fact that S dual bounded in $H^p(\mathbb{B})$ implies that the measure μ_S is Carleson [4] by use of the Wirtinger inequality. Trying to generalize this in the case of convex domains of finite type, I realize that this can be extended up to the case of the "strongly pseudo-convex domains" introduced in the abstract.

I have the feeling that this new notion goes beyond the fact that it is well suited for interpolating sequences ; it seems to capture all the domains which proved to be "tractable" for analysis in several complex variables. Moreover it may open the door for the study of domains which are not of finite type.

Let me be more precise about this. First in this work "domains" in $\mathbb{C}^n$ will always means bounded pseudo-convex domains with $\mathcal{C}^\infty$ smooth boundary.

Let W be the set of weakly pseudo-convex points of $\partial\Omega$, i.e. W is the zero set of the determinant of the Lévi form $\mathcal{L}$ of $\partial\Omega$. Let π be the normal projection from Ω onto $\partial\Omega$, defined in a neighbourhood $\mathcal{U}$ of $\partial\Omega$ in Ω.

Let $\alpha \in \partial\Omega$; by linear change of variables we can suppose that $\alpha = 0 \in \partial\Omega \subset \mathbb{C}^n,\ z_1 = 0$ is the equation of the complex tangent space. Extend the projection π locally near $0 \in \partial\Omega$ to a $\mathcal{C}^\infty$ diffeomorphism $\tilde{\pi}\ :\ \partial\Omega \longrightarrow T_0(\partial\Omega)$.

The condition for Ω to be strongly pseudo-convex means that there is a neighbourhood V_0 of 0 and a complex direction, say the z_n-axis, such that the slices

$$\tilde{\pi}(W \cap V_0) \cap \{z_1 = 0\} \cap \{z_2 = a_2\} \cap \cdots \cap \{z_{n-1} = a_{n-1}\}$$

have upper Minkowski dimension less than $2 - \beta,\ \beta = \beta_\alpha > 0$.

We shall show that the following domains are strongly pseudo-convex : domains of finite type in $\mathbb{C}^2$, convex domains of finite type in $\mathbb{C}^n$, domains with locally diagonalizable Lévi form and of finite type in $\mathbb{C}^n$ and domains in $\mathbb{C}^n$ with real analytic boundary.

But we have also domains not of finite type which are strongly pseudo-convex: domains like $\left|z_1\right|^2 + \exp\{1 - \left|z_2\right|^{-2}\} < 1$. The set of weakly pseudo-convex points is the manifold $W := \{z \in \mathbb{C}^2\ s.t.\ |z_1| = 1,\ z_2 = 0\}$, which is of real dimension 1, hence of Minkowski dimension $1 < 2$.

In order to state more precisely the first main result, we need the notion of a "good" family of polydiscs, directly inspirated by the work of Catlin [8].

Let $\alpha \in \partial\Omega$ and let $b(\alpha) = (L_1,\ L_2, ...,\ L_n)$ be an orthonormal basis of $\mathbb{C}^n$, such that $(L_2, ...,\ L_n)$ is a basis of the tangent complex space $T^{\mathbb{C}}_\alpha$ of $\partial\Omega$ at α hence L_1 is the complex normal at α to $\partial\Omega$. Let $m(\alpha) = (m_1,\ m_2, ...,\ m_n) \in \mathbb{N}^n$ be a multi-index at α with $m_1 = 1,\ \forall j \geq 2,\ m_j \geq 2$.

For $a\in\mathcal{U}$ such that $\pi(a)=\alpha$ and $\delta>0$ set $P_a(\delta):=\prod_{j=1}^n \delta D_j$, the polydisc such that δD_j is the disc centered at a, parallel to L_j with radius $\delta r(a)^{1/m_j}$, where $r(a)=d(a,\ \Omega^c)$ is the distance from a to the boundary of Ω.

This way we have a family of polydiscs $\mathcal{P}:=\{P_a(\delta)\}_{a\in\Omega}$ defined by the family of basis $\{b(\alpha)\}_{\alpha\in\partial\Omega}$, the family of multi-indices $\{m(\alpha)\}_{\alpha\in\partial\Omega}$ and the number δ.
Now we can set

Definition 1.1. *We say that $\mathcal{P}$ is a "good family" of polydiscs for Ω if the $m_j(\alpha)$ are uniformly bounded on $\partial\Omega$ and if it exists $\delta_0>0$ such that all the polydiscs $P_a(\delta_0)$ of $\mathcal{P}$ are contained in Ω. In this case we call $m(\alpha)$ the multi-type at α of the family $\mathcal{P}$.*

We notice that, for a good family $\mathcal{P}$, the multi-type is always finite.

We can see easily that there is always good families of polydiscs in a domain Ω in $\mathbb{C}^n$: for a point $\alpha\in\partial\Omega$, take any orthonormal basis $b(\alpha)=(L_1,\ L_2,...,\ L_n)$, with L_1 the complex normal direction, and the "minimal" multitype $m(\alpha)=(1,\ 2,...,\ 2)$. Then, because $\partial\Omega$ is of class $\mathcal{C}^2$ and compact, we have the existence of a uniform $\delta_0>0$ such that the family $\mathcal{P}$ is a good one. But as we shall see, this one is actually good just for the stricly pseudo-convex domains.

Now let Ω be a domain in $\mathbb{C}^n$ equipped with a good family of polydiscs $\mathcal{P}$; let S be sequence of points in Ω, we shall say that S is δ separated if the points of S are centers of disjoint polydiscs of $\mathcal{P}$, i.e. if:

$$\forall a\in S,\ \forall b\in S\backslash\{a\},\ P_a(\delta)\cap P_b(\delta)=\emptyset.$$

For $a\in\Omega$, let $\mu(a):=\displaystyle\sum_{j=2}^n\frac{1}{m_j(a)}$, where $m_j(a):=m_j(\pi(a))$ and $r(a):=d(a,\Omega^c)$ is the distance from a to the boundary. Now we have:

Theorem 1.2. *Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^n$. Let $\mathcal{P}=\{P_a(\delta_0)\}_{a\in\Omega}$ be a good family of polydiscs for Ω and S a δ-separated sequence of points such that $\pi(S)\subset W$, W the set of weakly pseudo-convex points of $\partial\Omega$. Then there is a constant M such that*

$$\sum_{a\in S} r(a)^{1+2\mu(a)}\leq M<\infty.$$

Here we already see that bigger are the $m_j(a)$ better is the inequality.

In order to continue, we shall need the following standard definition of the Blaschke class, a class of $(1,1)$ currents which are candidate to be supported by the zero set of a function in the Nevanlinna space.

Definition 1.3. *A holomorphic divisor X in the domain Ω is in the Blaschke class if there is a constant $C>0$ such that, with Θ its associated $(1,1)$ current,*

$$\forall\beta\in\Lambda^\infty_{n-2,\ n-2}(\overline{\Omega}),\ \left|\int_\Omega\rho(z)\Theta\wedge\partial\bar\partial\rho\wedge\beta\right|\leq C\left\|\beta\right\|_\infty, \tag{1.1}$$

where $\Lambda^\infty_{n-2,\ n-2}(\overline{\Omega})$ is the space of $(n-2,n-2)$ continuous form in $\overline{\Omega}$, equipped with the sup norm of the coefficients.

In particular $(1,1)$ current associated to the zero set of a function in the Nevanlinna class is in this Blaschke class and G. Henkin [16] and H. Skoda [25] proved, in the case of strictly pseudo-convex domains, that conversely if the current of integration of an analytic variety X is in the Blaschke class, then X is the zero set of a function in the Nevanlinna class.

In the case of convex domains of strict finite type in $\mathbb{C}^n$, J. Bruna, P. Charpentier and Y. Dupain [7] proved the characterization of the zero sets of the Nevanlinna class. The characterization for all convex domains of finite type in $\mathbb{C}^n$ was obtained by A. Cumenge [11] by a very original method and, by a more classical one, by K. Diederich and E. Mazzilli [14].

The condition find in theorem 1.2 is strongly reminiscent of the Malliavin condition which was used by H. Skoda [25], and in fact, with the help of the Wirtinger inequality, we get the following generalization of it:

Theorem 1.4. *Let X be a divisor in the Blaschke class of a domain Ω. Suppose that Ω is equipped with a minimal family of polydiscs $\mathcal{P}$ and let S be a δ-separated sequence of points in X. There exists a constant $C > 0$ such that*

$$c_{n-1}\delta^{2n-2}\sum_{a\in S} r(a)^n \leq C < \infty,$$

where c_{n-1} is the volume of the unit ball of $\mathbb{C}^{n-1}$.

This is precisely the inequality in theorem 1.2 with the "minimal" multi-type $m = (1, 2, ..., 2)$.
Theorem 1.2 and theorem 1.4 together gives the main result of this first part :

Theorem 1.5. *Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^n$. Let $\mathcal{P} = \{P_a(\delta_0)\}_{a\in\Omega}$ be a good family of polydiscs for Ω and S a δ-separated sequence of points contained in a divisor X of the Blaschke class of Ω. Then there is a constant M such that*

$$\sum_{a\in S} r(a)^{1+2\mu(a)} \leq M < \infty,$$

where $r(a)$ is the distance from a to the boundary of Ω and $\mu(a) := \sum_{j=2}^{n} \frac{1}{m_j(a)}$, with $(1,\ m_2(a), ..., m_n(a))$ is the multi-type associated to the family $\mathcal{P}$.

Theorem 1.5 says precisely that the measure $\mu_S := \sum_{a\in S} r(a)^{1+2\mu(a)}$ associated to the sequence S is finite. This generalise the result [4] we got for the unit ball in $\mathbb{C}^n$.

The idea behind the proof of this result is to separate the "bad points", namely the points which are not stricly pseudo-convex, from the "good ones". We use the fact that the bad ones are few numerous enough to have directly for all of them the right inequality. For the good ones we use theorem 1.4 which generalizes the Malliavin condition.

In a second part we introduce the notion of p-regularity of a domain Ω. Define, as usual, the Hardy class $H^p(\Omega)$ as the closure in $L^p(\partial\Omega,\ \sigma)$ of polynomials where σ is the Lebesgue measure on $\partial\Omega$. The Hilbert space $H^2(\Omega)$ has reproducing kernels (or Szegö kernels) $k_a(z) \in H^2(\Omega)$. If $\mathcal{P}$ is a good family of polydiscs for Ω, then we remark that $P_a(2)$ overflows Ω, because the radius of the polydisc centered at a is $2r(a)$ in the real normal direction, so $P_a(2)$ cannot be contained in Ω.

Definition 1.6. *We shall say that Ω is p-regular with respect to the family $\mathcal{P}$ if :*

$$\exists C > 0 \ s.t.\ \forall a \in \Omega,\ \sigma(\partial\Omega \cap P_a(2)) \leq C \left\|k_a\right\|_{p'}^{-1},$$

where p' is the conjugate exponent of p.

In an analogous way we define the geometric Carleson measures, where $\Omega \cap P_a(2)$ is the Carleson window of center a :

Definition 1.7. *Let μ be a borelian positive measure on Ω. We shall say that μ is a geometric Carleson measure in Ω if :*

$$\exists C>0 \ s.t. \ \forall a\in\Omega,\ \mu(\Omega\cap P_a(2))\leq C\sigma(\partial\Omega\cap P_a(2)).$$

Before stating our second result, we need yet another definition.

Definition 1.8. *We shall say that the sequence S of points in Ω is dual bounded in $H^p(\Omega)$ if there is a bounded sequence of elements in $H^p(\Omega)$, $\{\rho_a\}_{a\in S}\subset H^p(\Omega)$ which dualizes the associated sequence of reproducing kernels, i.e.*

$$\exists C>0 \ s.t. \ \forall a\in S,\ \left\|\rho_a\right\|_p\leq C,\ \forall a,b\in S,\ \langle\rho_a,\ k_b\rangle=\delta_{a,b}\left\|k_b\right\|_{p'}.$$

Now we have

Theorem 1.9. *Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^n$ with a good family $\mathcal{P}$ of polydiscs and which is p-regular with respect to $\mathcal{P}$. If the sequence of points $S\subset\Omega$ is dual bounded in $H^p(\Omega)$, then the measure $\nu:=\sum_{a\in S}r(a)^{1+2\mu(a)}\delta_a$, with $\mu(a):=\sum_{j=2}^n\frac{1}{m_j(a)}$, is a geometric Carleson measure in Ω.*

To prove this theorem a key ingredient is theorem 1.5.

The dual bounded sequences are related to $H^p(\Omega)$ interpolating sequences:

Definition 1.10. *We say that the sequence S of points in Ω is $H^p(\Omega)$ interpolating if*

$$\forall\lambda\in\ell^p(S),\ \exists f\in H^p(\Omega) \ s.t. \ \forall a\in S,\ f(a)=\lambda_a\left\|k_a\right\|_{p'}.$$

Clearly if S is $H^p(\Omega)$ interpolating then S is dual bounded in $H^p(\Omega)$: just interpolate the basic sequence of $\ell^p(S)$. We say that S has the linear extension property if S is $H^p(\Omega)$ interpolating and if moreover there is a bounded linear operator $E\ :\ \ell^p(S)\longrightarrow H^p(\Omega)$ making the interpolation.

As an application of theorem 1.9 we prove:

Theorem 1.11. *If Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $S\subset\Omega$ is a dual bounded sequence of points in $H^p(\Omega)$, then, for any $q<p$, S is $H^q(\Omega)$ interpolating with the linear extension property, provided that $p=\infty$ or $p\leq 2$.*

The proof of this theorem relies heavily on results of McNeal and Stein ([21], [22], [19]) and is an application of [2].

In the third part, we give examples of strongly pseudo-convex domains, namely the pseudo-convex domains of finite type in $\mathbb{C}^2$, the convex domains of finite type in $\mathbb{C}^n$, the domains locally diagonalizable of finite type in $\mathbb{C}^n$, and domains with real analytic boundary.

At this point we have to notice that domains of finite type in D'Angelo sense [12] have a finite Catlin multi-type [8] and hence a finite linear multi-type (see section 2.2). In general the multi-type is strictly better than the linear multi-type, but this is not always the case; for convex domains they are the same [19], [6], [29].

Let $\mathcal{L}$ be the Lévi form of $\partial\Omega$ and set $\mathcal{D}:=\det\mathcal{L}$; the set W of points of weak pseudo-convexity is $W:=\{z\in\partial\Omega \ s.t. \ \mathcal{D}(z)=0\}$. The proof for showing the examples is based on the following theorem.

Theorem 1.12. *Let Ω be a domain in $\mathbb{C}^n$ of finite linear type, and $\mathcal{D}$ the determinant of its Lévi form. Suppose that:*

$$\forall\alpha\in\partial\Omega,\ \exists v\in T_\alpha^{\mathbb{C}}(\partial\Omega) \ s.t. \ \exists k\in\mathbb{N},\ \frac{\partial^k\mathcal{D}}{\partial v^k}(\alpha)\neq 0,$$

then Ω is strongly pseudo-convex and can be equipped with a family of polydiscs whose multi-type is the given linear type.

The proof of this theorem uses the differentiable preparation theorem of Malgrange [27] to reduce locally to the case of the zeros of polynomials. Then we use a theorem of Ostrowski [24] on the regularity of roots of polynomial equations which allows us to estimate the Minkowki dimension of the weakly pseudo-convex points.

The case of the pseudo-convex domains of finite type in $\mathbb{C}^2$ is a simple application of the theorem 1.12 because the Lévi form is just a function and the finite type hypothesis applies directly. It can also be seen as a special case of domains locally diagonalizable.

The case of the convex domains of finite type in $\mathbb{C}^n$ is more delicate and to prove it we have to use, beside theorem 1.12, the maximum principle for the Monge-Ampère operator.

The case of domains locally diagonalizable of finite type in $\mathbb{C}^n$ reduces quite easily to the theorem 1.12

Bounded domains with real analytic boundary are automatically of finite type [13] and reduces to theorem 1.12.

Finally in the appendix we define the upper Minkowski dimension of a subset of $\mathbb{R}^n$ and state the properties we shall use about it.

Since the presentation of this work in june 2007 at the C.I.R.M. the case of domains with a locally diagonalizable Lévi form and the case of domains with real analytic boundary were added. The application to interpolating sequences in convex domains of finite type in $\mathbb{C}^n$ is also added.

2. Strongly pseudo-convex domains.

2.1. **Examples of domains with a good family of polydiscs.** The stricly pseudo-convex domains in $\mathbb{C}^n$: they have a good family of polydiscs associated with the best possible multi-type, the one defined by Catlin [8], $\forall a \in \mathcal{U},\ m_1 = 1,\ \ \forall j = 2, ..., n,\ m_j(a) = 2$. Moreover these polydiscs are associated to the pseudo balls of a structure of spaces of homogeneous type (Korany-Vagi [17], Coifman-Weiss [10]).

The finite type domains in $\mathbb{C}^2$: also here we have the best possible multi-type and a structure of spaces of homogeneous type. (Nagel-Rosay-Stein-Wainger [23])

The convex finite type domains in $\mathbb{C}^n$: again we have the best multi-type and a structure of spaces of homogeneous type. (McNeal [20])

2.2. **Linear finite type.** In order to prove what we need in this section ,we have to recall precisely the definition of the multi-type [8] and the linear multi-type [19], [29]. We shall take the definitions and the notations from J. Yu [29].

Let Ω be a domain in $\mathbb{C}^n$ defined by the function ρ, and let $p \in \partial\Omega$ be fixed.

Let Γ_n be the set of the n-tuples of integers $\Lambda = (m_1, ..., m_n)$ with $1 \leq m_j \leq \infty$ and such that :
(i) $m_1 \leq m_2 \leq \cdots \leq m_n$.
(ii) for all $k = 1, ..., n$, either $m_k = +\infty$ or there are strictly positive integers $a_1, ..., a_k$ such that $\displaystyle\sum_{j=1}^{k} a_j/m_j = 1$.

An element in Γ_n will be called a *weight*. The set Γ_n of weights can be ordered lexicographically: $\Lambda = (m_1, ..., m_n) < \Lambda' = (m'_1, ..., m'_n)$ if there is a k such that $\forall j < k,\ m_j = m'_j$ and $m_k < m'_k$.

A weight is said to be *distinguished* if there exist holomorphic coordinates $z_1, ..., z_n$, in a neighbourhood of p with p mapped to the origine and such that :

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{m_i} < 1 \Longrightarrow \partial^{\alpha}\overline{\partial}^{\beta}\rho(p) = 0, \tag{2.1}$$

where $\partial^{\alpha} := \dfrac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$ and $\overline{\partial}^{\beta} := \dfrac{\partial^{|\beta|}}{\partial \overline{z}_1^{\beta_1} \cdots \partial \overline{z}_n^{\beta_n}}$.

Definition 2.1. *The multi-type $\mathcal{M}(\partial\Omega, p)$ is the smallest weight $\mathcal{M} := (m_1, ..., m_n)$ in Γ_n (in lexicographic sense) such that $\mathcal{M} \geq \Lambda$ for every distinguished weight Λ.*

Because $\partial\Omega$ is smooth at p, we always have $m_1 = 1$.

We call a weight Λ *linearly distinguished* if there exists a complex linear change of variables near p with p mapped to the origin and such that (2.1) holds in these new coordinates.

Definition 2.2. *The linear multitype $\mathcal{L}(\partial\Omega,\ p)$ is the smallest weight $\mathcal{L} := (m_1, ..., m_n)$ such that $\mathcal{L} \geq \Lambda$ for every linear distinguished weight Λ. We shall say that Ω is of linear finite type if*
$\exists m \in \mathbb{N}$ *s.t.* $\forall p \in \partial\Omega,\ \mathcal{L}(\partial\Omega,\ p) \leq (m, ..., m)$.

Clearly we have $\mathcal{L}(\partial\Omega,\ p) \leq \mathcal{M}(\partial\Omega,\ p)$.

If Ω is of linear finite type $\mathcal{L}(\partial\Omega,\ p) = (m_1, ..., m_n)$, then for $p \in \partial\Omega$ fixed, there is a $\mathbb{C}$-linear change of variables such that :

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{m_i} < 1 \Longrightarrow \partial^{\alpha}\overline{\partial}^{\beta}\rho(p) = 0. \tag{2.2}$$

Back to the previous coordinates, this means that there are complex directions $v_1,\ v_2, ..., v_n$ with v_1 the complex normal at p, such that:

$$\sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{m_i} < 1 \Longrightarrow \partial_v^{\alpha}\overline{\partial}_v^{\beta}\rho(p) = 0, \tag{2.3}$$

with now $\partial_v^{\alpha} := \dfrac{\partial^{|\alpha|}}{\partial v_1^{\alpha_1} \cdots \partial v_n^{\alpha_n}}$ and $\overline{\partial}_v^{\beta} := \dfrac{\partial^{|\beta|}}{\partial \overline{v}_1^{\beta_1} \cdots \partial \overline{v}_n^{\beta_n}}$ are the derivatives in the directions v_j.

The aim of this section is to show

Theorem 2.3. *If Ω is a domain in $\mathbb{C}^n$ of finite linear type, then there is a good family $\mathcal{P}$ of polydiscs such that the multitype associated to $\mathcal{P}$ is precisely the linear multitype of Ω.*

This theorem will be a simple consequence of the following two lemmas.

Lemma 2.4. *Let $k > 1$; all derivatives of order less to m_k in real directions belonging to the space* $\mathrm{Span}\{v_k,\ v_{k+1}, ..., v_n\}$ *generated by $v_j,\ j \geq k$, are zeros.*

This means that the order of contact of those lines with $\partial\Omega$ at p is bigger than m_k.

Proof.
This property being invariant by linear change of variables, we can suppose that $p = 0$ and the v_j are the z_j. So we have (2.1). Let x a real direction in $\mathrm{Span}\{z_k,\ z_{k+1}, ..., z_n\}$, for instance

$$\frac{\partial}{\partial x}=\sum_{j=k}^{n}a_j\frac{\partial}{\partial z_j}+\sum_{j=k}^{n}b_j\frac{\partial}{\partial \overline{z}_j}.$$

If $(\frac{\partial}{\partial x})^m\rho(0)\neq 0$ for a $m<m_k$ then

$$\left(\sum_{j=k}^{n}a_j\frac{\partial}{\partial z_j}+\sum_{j=k}^{n}b_j\frac{\partial}{\partial \overline{z}_j}\right)^m\rho(0)\neq 0\ ; \tag{2.4}$$

let us take a mixted term of this symbolic power: $\frac{\partial^{r+s}\rho}{\partial z_l^r\partial \overline{z}_j^s}(0)$ with $r+s=m$. We set $\alpha_l:=r,\ \beta_j:=s$ and all other terms zero. Because $m_l\geq m_k,\ m_j\geq m_k$, we get

$$\frac{\alpha_l}{m_l}+\frac{\beta_j}{m_j}\leq\frac{r}{m_k}+\frac{s}{m_k}=\frac{r+s}{m_k}=\frac{m}{m_k}<1,$$

hence $\frac{\partial^{r+s}\rho}{\partial z_l^r\partial \overline{z}_j^s}(0)=0$.

The same for the "pure" term: $\frac{\partial^{r+s}\rho}{\partial z_l^r\partial z_j^s}(0)=0$ still with $r+s=m$ and setting $\alpha_l:=r,\ \alpha_j:=s$. Hence we reach a contradiction with (2.4), which gives the lemma. □

In the complex space generated by $v_n,\ v_{n-1}$ we take $e_n(p):=v_n$ and $e_{n-1}(p)$ the complex direction orthogonal to v_n.

In the complex space generated by $v_n,\ v_{n-1},\ v_{n-2}$ we take $e_{n-2}(p)$ the complex direction orthogonal to $(e_n,\ e_{n-1})$, etc ... up to $e_2(p)$. We set $e_1(p)$ the normal vector at p to $\partial\Omega$.

This built, à la Gram-Schmitt, an orthonormal basis of $T_p^{\mathbb{C}}$.

This is the way to define the associated polydiscs. Let $p\in\partial\Omega$ and a point $a\in\Omega$ on the real normal at p to $\partial\Omega$, i.e. $\pi(a)=p$.

Let the polydisc centered at a and such that : $P_a(\delta):=\prod_{j=1}^n\delta D_j$ with D_j the disc in the direction e_j, of center a and radius $r(a)^{1/m_j(p)}$.

Lemma 2.5. *There exists $\delta_0>0$ such that $\forall a\in\mathcal{U},\ \forall\delta\leq\delta_0,\ P_a(\delta)\subset\Omega$.*

Proof.
Let $a\in\mathcal{U}$, where $\mathcal{U}$ is the neighbourhood of $\partial\Omega$ in Ω where the projection π on $\partial\Omega$ is defined, and let $p=\pi(a)\in\partial\Omega$. Let us see the Taylor formula of ρ at p, only in the directions $e_n(p)$ and $e_{n-1}(p)$. Set, to have simpler notations, $k=m_{n-1},\ l=m_n\Longrightarrow l>k$ the associated types and $y,\ x$ the corresponding real variables. We get, the other variables set to 0 and $p=0$:

$$\rho=-z+Ay^k+B(x\cos\theta+y\sin\theta)^l+o((|x|+|y|)^l).$$

Of course $A\neq 0,\ \cos\theta\neq 0$. We want to see the "rectangle" centered at $a=(0,0,r(a))$ and of sides parallel to the axes, we can enclosed in $\rho\leq\rho(a)$.

Let $|y|\leq\delta r(a)^{1/k}$ and $|x|\leq\delta r(a)^{1/l}$; we get

$$\left|Ay^k+B(x\cos\theta+y\sin\theta)^l\right|\leq|A|\,|y|^k+|B|\left(\delta r(a)^{1/l}\,|\cos\theta|+\delta r(a)^{1/k}\right)^l\leq$$
$$\leq\left(|A|\,\delta^k+|B|\,(|\cos\theta|+|\sin\theta|)^l\delta^l\right)r(a),$$

because $l>k\Longrightarrow r(a)^{1/k}\leq r(a)^{1/l}$.

We take δ_0 such that $\left(|A|\,\delta^k+|B|\,(|\cos\theta|+|\sin\theta|)^l\delta^l\right)<1/2$, which is always possible with δ_0 uniformly (with respect to $a\in\Omega$) small, because the constants $A,\ B$ being derivatives of ρ are uniformly bounded.

We continue the same way by adding $e_{n-2}(p)$, etc ... and the lemma after a finite number of steps. $\square$

Clearly the last lemma gives the theorem by taking the family $\mathcal{P} := \{P_a(\delta_0),\ a \in \Omega\}$ with $P_a(\delta_0) := \prod_{j=1}^n \delta_0 D_j$ and D_j the disc in the direction e_j, of center a and radius $r(a)^{1/m_j(p)}$.

2.3. **Sequences projecting on weak pseudo-convex points.**

Let $W \subset \partial\Omega$ be the set of weakly pseudo-convex points of a domain Ω.

Theorem 2.6. *let $\mathcal{P}$ be a good family of polydiscs on a strongly pseudo-convex domain Ω in $\mathbb{C}^n$, and S be a δ-separated sequence of points in Ω. If $\pi(S) \subset W$ then we have :*

$$\sum_{a\in S} r(a)^{1+2\mu(a)} \leq M < \infty, \tag{2.5}$$

where $\mu(a) := \displaystyle\sum_{j=2}^{n} \frac{1}{m_j(a)}$, $(1,\ m_2(a), ..., m_n(a))$ *being the multi-type of the point* $\pi(a)$.

Proof.
The polydisc $P_a(\delta)$ has radius $\gamma := \delta r(a)$ in the normal direction and in the conjugate to the normal direction and has radii $(\gamma^{1/m_2(a)}, ..., \gamma^{1/m_n(a)})$ in the complex tangent directions. Let us denote $L_2, ...,\ L_n$ the complex tangent directions associated to $\alpha = \pi(a)$ with multi-type $(m_2(a), ..., m_n(a))$.

Now fix $\alpha \in W \subset \partial\Omega$ and let V_α be a neighbourhood of α in $\partial\Omega$ such that $\tilde{\pi}$ is a diffeomorphism from V_α on a neighbourhood of α on the tangent space T_α.
The complex tangent line $\mathbb{C}L_j,\ j = 2, ...,\ n$, is a subspace of the tangent space T_α hence $V_\alpha^j := \tilde{\pi}^{-1}(\mathbb{C}L_j)$ is a smooth submanifold of real dimension 2 of $V_\alpha \subset \partial\Omega$.

Because Ω is strongly pseudo-convex, we know that there is a complex direction in the complex tangent space at α, say z_n, along which W is of upper Minkowski dimension $2 - \beta,\ \beta > 0$.
Let $S \subset \Omega$ *s.t.* $\pi(S) \subset W$ be the δ-separated given sequence. By the compacity of $\partial\Omega$, without loss of generality, we may suppose that $S \subset \pi^{-1}(V_\alpha)$.

The proof will follow from several reductions.

2.3.1. *Reduction to a layer parallel to the complex space.* By use of the $\mathcal{C}^\infty$ diffeomorphism $\tilde{\pi}$, we can suppose that $\partial\Omega \simeq T_0(\partial\Omega)$ near $\alpha = 0$, the complex tangent space is $z_1 = 0$, the real one is $x_1 := \Re z_1 = 0$.

Let C_γ be a layer parallel to $\partial\Omega$ at a distance γ from the boundary, i.e.
$a = (a_1, ...,\ a_n) \in C_\gamma \Longleftrightarrow \Re a_1 \simeq r(a) \simeq \gamma.$
Now let $S_\gamma := S \cap C_\gamma$ *s.t.* $\pi(a) \in V_\alpha$. We refine the layer by fixing the imaginary part of $z_1\ :\ \Im z_1 \simeq t$, and we set $S_{\gamma,t} := S \cap C_{\gamma,t}$, with $a \in C_{\gamma,t} \Longleftrightarrow \Re a_1 \simeq r(a) \simeq \gamma,\ \Im a_1 \simeq t$.

2.3.2. *Reduction to a fixed multi-type.* There is only a finite set of possible multi-types for the points of S because we have a good family of polydiscs. Hence it is enough to show the inequality 2.5 for the points $a \in S$ with a fixed multi-type, $m(a) = (1,\ m_2, ...,\ m_n)$.
Now we can apply the lemma 5.5 from the appendix to the sequence $S_{\gamma,t}$; because $m_2 \leq \cdots \leq m_n$, we set :

$$r := \gamma^{1/m_2},\ l := \gamma^{\frac{1}{m_n} - \frac{1}{m_2}}.$$

The lemma gives $\displaystyle\sum_{a\in S_{\gamma,t}} \text{Area}(P_a) \leq Cl^\beta r^\beta = C\gamma^{\beta/m_n}$. But $\text{Area}(P_a) = \gamma^{2\mu(a)}$ hence

$$\sum_{a\in S_{\gamma,t}} \text{Area}(P_a) = \sum_{a\in S_{\gamma,t}} \gamma^{2\mu(a)} = \sum_{a\in S_{\gamma,t}} r(a)^{2\mu(a)} \leq C\gamma^{\beta/m_n}.$$

2.3.3. *Adding along the conjugate to the normal.* For the previous sequence, the imaginary part of a_1 was fixed at t. Because the sequence is δ-separated, the distance between points on the conjugate to the normal is $\delta\gamma$ hence there are essentially $1/\delta\gamma$ such points. So we get

$$\sum_{a\in S_\gamma} r(a)^{1+2\mu(a)} \simeq \frac{1}{\delta\gamma} \sum_{a\in S_{\gamma,t}} \gamma^{1+2\mu(a)} = \frac{1}{\delta} \sum_{a\in S_{\gamma,t}} \gamma^{2\mu(a)} \leq \frac{C}{\delta}\gamma^{\beta/m_n}.$$

2.3.4. *Adding the layers.* Because the sequence is separated, the layers can be ordered this way $\gamma_k = \nu^k\gamma_0,\ k\in\mathbb{N}$, where γ_0 is the farthest point from the boundary and $\nu = \dfrac{1-\delta}{1+\delta}$.
We have to add them and because of the previous inequality we get

$$\sum_{k\in\mathbb{N}} \sum_{a\in S_{\nu^k}} r(a)^{1+2\mu(a)} \leq \frac{C}{\delta} \sum_{k\in\mathbb{N}} \nu^{\beta k/m_n} \simeq \frac{m_n}{\beta\delta^2} < \infty.$$

2.3.5. *Adding for all the multi-types.* We have a finite sum of finite numbers so it is finite, and the theorem with

$$M \lesssim \frac{m_n}{\beta\delta^2}.$$

□

2.4. **Separated sequences in divisors of the Blaschke class.**

Let Ω be a domain in $\mathbb{C}^n$ defined by the function ρ.

We shall give here a simple generalization of the Malliavin condition fullfilled by the zeros of a function in the Nevanlinna class $\mathcal{N}(\Omega)$ of Ω.

Let $u\in\mathcal{N}(\Omega)$, this means that u is holomorphic in Ω and has the estimate

$$\|u\|_{\mathcal{N}} = \sup_{\epsilon>0} \int_{\partial\Omega_\epsilon} \log^+ |u(\zeta)|\ d\sigma(\zeta) < \infty,$$

with $\forall\epsilon > 0,\ \Omega_\epsilon := \{z\in\Omega\ s.t.\ \rho(z) < -\epsilon\}$.

Now let $X := u^{-1}(0)$; put $\Theta := \partial\bar\partial\log|u|$ the $(1,1)$ current associated to X. Let us recall quickly how Green formula gives us the Blaschke condition [25] :

$$\log|u(a)| = \int_{\partial\Omega} \log|u(\zeta)|\, P(a,\zeta)\, d\sigma(\zeta) + \int_\Omega \Delta\log|u(z)|\, G(a,z)\, dm(z),$$

where $a\in\Omega,\ G(a,z)$ is the Green kernel of Ω with pole at a and $P(a,\zeta)$ is the Poisson kernel of Ω.

Let $a\in\Omega$ fixed such that $u(a)\neq 0$ and we suppose u normalized to have

$$u(a) = 1. \tag{2.6}$$

Since $G\leq 0,\ 0\leq P(a,\zeta)\leq c$, we get

$$-\int_\Omega \Delta\log|u(z)|\, G(a,z)\, dm(z) = \int_{\partial\Omega} \log^+|u|\, P(a,\zeta)\, d\sigma - \int_{\partial\Omega} \log^-|u|\, P(a,\zeta)\, d\sigma,$$

$$-\int_\Omega \Delta\log|u(z)|\, G(a,z)\, dm(z) \leq \int_{\partial\Omega} \log^+|u(\zeta)|\, P(a,\zeta)\, d\sigma \leq$$
$$\leq c\int_{\partial\Omega} \log^+|u(\zeta)|\ d\sigma(\zeta) \leq c\,\|u\|_{\mathcal{N}}.$$

But $\Delta \log |u(z)| = \mathrm{Tr}\Theta$, the trace of Θ, and Θ is a positive current, hence its trace controls all coefficients. On the other hand we have $-G(a,z) \simeq -\rho(z)$ near $\partial\Omega$, hence we get the classical Blaschke necessary condition :

$$0 \leq \int_\Omega -\rho(z)\mathrm{Tr}\Theta(z) \leq B := c\,\|u\|_{\mathcal{N}}\,.$$

This implies that, if β is a $(n-2,n-2)$ continuous form in $\overline{\Omega}$, because the coefficients of Θ are controlled by $\mathrm{Tr}\Theta$:

$$\left|\int_\Omega \rho(z)\Theta\wedge\partial\overline{\partial}\rho\wedge\beta\right| \leq c\,\|\beta\|_\infty\,\|u\|_{\mathcal{N}}\,. \tag{2.7}$$

This will be our definition of the Blaschke class, as in the definition 1.3 from the introduction.

In particular the zero set of a function in the Nevanlinna class is in the Blaschke class.

For $n=2,\ \beta=1$, this gives the Malliavin condition([25], p 277, II.2.6) because, by Stokes formula:

$$\int_\Omega i\partial\rho\wedge\overline{\partial}\rho\wedge\Theta = \int_\Omega -\rho(i\partial\overline{\partial}\rho)\wedge\Theta < \infty,$$

hence the projection of X on the complex normal has finite area. This is what we shall generalize to $n>2$.

We shall prove this generalization locally : let us suppose that $0\in\partial\Omega$, that the vector L_1 on the z_1 axis, is the complex normal at 0 to $\partial\Omega$, and hence the vectors L_j of the z_j axis $2\leq j\leq n$, make an orthonormal basis of the complex tangent space at 0. For z in a neighbourhood of 0, the complex line $\{z+\lambda L_1,\ \lambda\in\mathbb{C}\}$ is still transversal to $\partial\Omega$. Let $\mathcal{V}$ be such a neighbourhood; let us set $\gamma := i\sum_{j=1}^n \alpha_j(z_j,\overline{z}_j)\,dz_j\wedge d\overline{z}_j$, with $\mathbb{D}$ the unit disc in $\mathbb{C}$ and $\alpha_j\in\mathcal{C}^\infty(\mathbb{D}),\ 0\leq\alpha_j\leq 1$, with compact support and the $\alpha_j=1$ near 0.

Now we have that $\partial\gamma=\overline{\partial}\gamma=0$ and γ is a positive $(1,1)$ form. Set $\beta:=\gamma^{\wedge(n-2)}$ and apply Stokes formula to $\rho\Theta\wedge\overline{\partial}\rho\wedge\beta$:

$$0=\int_{\partial\Omega}\rho\Theta\wedge\overline{\partial}\rho\wedge\beta=\int_\Omega\Theta\wedge\partial\rho\wedge\overline{\partial}\rho\wedge\beta-\int_\Omega\rho\Theta\wedge\partial\overline{\partial}\rho\wedge\beta,$$

because Θ and β are closed. Hence

$$\left|\int_\Omega\Theta\wedge\partial\rho\wedge\overline{\partial}\rho\wedge\beta\right|=\left|\int_\Omega\rho\Theta\wedge\partial\overline{\partial}\rho\wedge\beta\right|<\infty,$$

because of the inequality (2.7).

Set $\mathcal{V}' := \bigcap_{j=1}^n\{\alpha_j=1\}\cap\mathcal{V}\cap\Omega$, because $\Theta\wedge\partial\rho\wedge\overline{\partial}\rho\wedge\beta$ is positive, we still have

$$\int_{\mathcal{V}'}\Theta\wedge\partial\rho\wedge\overline{\partial}\rho\wedge\beta<\infty.$$

But in this integral $\beta=\gamma^{\wedge(n-2)}$ with $\gamma=i\sum_{j=1}^n dz_j\wedge d\overline{z}_j$ and $\partial\rho\wedge\overline{\partial}\rho$ is the complex normal, hence $\int_{\mathcal{V}'}\Theta\wedge\partial\rho\wedge\overline{\partial}\rho\wedge\beta$ is equivalent to the sum of the areas of the projections of $X\cap\mathcal{V}'$ on the coordinate subspaces containing the complex normal([18], Proposition 2.48, p 55).

Because $\partial\Omega$ is compact, we have shown the theorem:

Theorem 2.7. *Let X be a divisor in the Blaschke class of a domain Ω. Then the sum of the areas of the projection of X, counting multiplicities, on the coordinate subspaces containing the complex normal is bounded.*

Now let Ω be a domain in $\mathbb{C}^n$. Let $a \in \mathcal{U},\ \alpha = \pi(a)$ and $P_a(\delta)$ the polydisc of the good family associated to Ω, with sides parallel to the basis at $\alpha,\ (L_1, ..., L_n)$, and L_1 the complex normal to $\partial\Omega$. One can take for instance the "minimal" good family associated to the multi-type $(1,\ 2,\ ...,\ 2)$.

Let $\mathbb{D}^n$ be the unit polydisc in $\mathbb{C}^n$, and let Φ_a be the bi-holomorphic application from $\mathbb{D}^n$ onto $P_a(\delta)$:

$$\forall z = (z_1, ..., z_n) \in \mathbb{D}^n,\ 1 \leq j \leq n,\ Z_j = a_j + \delta r(a)^{1/m_j(a)} z_j L_j.$$

If X is the zero set of a holomorphic function in Ω with $a \in X$, we can lift $X \cap P_a(\delta)$ in $\mathbb{D}^n$ by Φ_a^{-1}. Set $Y_a := \Phi_a^{-1}(X \cap P_a(\delta))$; we get

$$\mathrm{Area}(Y_a) = \sum_{j=1}^{n} A_j(Y_a),$$

where $A_j(Y_a)$ is the area of the projection of Y, counted with multiplicity, on the subspace orthogonal to the z_j axis.

On the other hand the Wirtinger inequality gives

$$c_{n-1} \leq \mathrm{Area}(Y_a),$$

where c_{n-1} is the area of the unit ball of $\mathbb{C}^{n-1}$. The change of variables formula allows us to lift these results on X and $P_a(\delta)$. Set $X_a := X \cap P_a(\delta),\ A_j(X_a)$ the area of the projection of X_a on the orthogonal to $L_j,\ j = 1, ...,\ n$. Recall that the multi-type is such that $m_1 = 1$. We have

Lemma 2.8. *(i)* $\mathrm{Area}(X_a) = \displaystyle\sum_{j=1}^{n} A_j(X_a)$.

(ii) $\forall j = 1, ..., n,\ A_j(X_a) = \delta^{2n-2} r(a)^{2\mu_j(a)} A_j(Y_a)$, *with* $\mu_j(a) = \displaystyle\sum_{k \neq j} \frac{1}{m_k(a)}$.

(iii) $\forall j = 2, ..., n,\ \mu_j(a) \leq n/2$; $\mu_1(a) \leq (n-1)/2$.

(iv) $c_{n-1} \leq \mathrm{Area}(Y_a)$.

Proof.

The *(i)* is classical([18], Proposition 2.48, p 55).

The application Φ_a sends $E_k :=$ {the orthogonal to the z_k axis} in $F_k :=$ {the orthogonal to L_k axis} and the jacobian of this restriction at the point a, $J_k\Phi$, is $J_k\Phi = \delta^{n-1} r(a)^{\mu_k(a)}$. Because the application is holomorphic, we get that the jacobian for the change of real variables is

$$|J_k|^2 = \delta^{2n-2} r(a)^{2\mu_k(a)},$$

which gives the *(ii)*.

For the *(iii)* we notice that

$$2 \leq j, k \leq n,\ m_k(a) \geq 2 \Longrightarrow \frac{1}{m_k(a)} \leq \frac{1}{2} \Longrightarrow \sum_{k \neq j,\ 2 \leq j,k \leq n} \frac{1}{m_k(a)} \leq \frac{n-2}{2}.$$

Hence if $2 \leq j \leq n,\ \mu_j(a) = \displaystyle\sum_{k \neq j,\ 2 \leq j,k \leq n} \frac{1}{m_k(a)} + 1 \leq n/2$;

if $j = 1,\ \mu_1(a) \leq \dfrac{n-1}{2} \leq n/2$. Hence *(iii)*.

The *(iv)* is just the Wirtinger inequality [15], hence the lemma. □

We can deduce the following corollary, which has its stronger form for S being a separated sequence for a family $\mathcal{P}$ with the minimal multi-type.

Corollary 2.9. *Let S be a δ-separated sequence of points in the Blaschke divisor X. There exist a constant $C > 0$ such that*

$$c_{n-1}\delta^{2n-2}\sum_{a\in S} r(a)^n \leq C,$$

where c_{n-1} *is the volume of the unit ball of* $\mathbb{C}^{n-1}$.

Proof.
Let $a\in X\cap\mathcal{U},\ \alpha=\pi(a)$ and $P_a(\delta)$ the associated polydisc ; the normal direction at α is noted L_1 and the complex ones $L_j,\ 2\leq j\leq n$. Let $1\leq j\leq n$ fixed, the area of the projection of $X_a := X\cap P_a(\delta)$ on $L_j^{\perp}$ is denoted $A_j(X_a)$.

Now let S be a δ-separated sequence of points on X.
We have

$$\forall a\in S,\ \forall j=1,...,n,\ A_j(X_a)\geq \delta^{2n-2}r(a)^{2\mu_j(a)}A_j(Y_a), \tag{2.8}$$

But the theorem 2.7 gives, for X is in the Blaschke class, because the X_a are disjoint :

$$\sum_{a\in S}\sum_{j=2}^{n} A_j(X_a)\leq M. \tag{2.9}$$

The Blaschke condition always gives :

$$\sum_{a\in S} r(a)\mathrm{Area}(X_a)\leq B \Longrightarrow \sum_{a\in S} r(a)A_1(X_a)\leq B. \tag{2.10}$$

Hence, using (iii) of the lemma, because $\forall j=2,...,n,\ \mu_j(a)\leq n/2$, and equation (2.9), we get

$$\forall a\in S,\ \forall j=2,...,n,\ A_j(X_a)\geq \delta^{2n-2}r(a)^n A_j(Y_a),$$

and, because $\mu_1(a)\leq (n-1)/2$,

$$\forall a\in S,\ A_1(X_a)\geq \delta^{2n-2}r(a)^{n-1}A_1(Y_a),$$

so

$$\forall a\in S,\ r(a)A_1(X_a)\geq \delta^{2n-2}r(a)^n A_1(Y_a).$$

Finally

$$\delta^{2n-2}r(a)^n\sum_{j=1}^{n} A_j(Y_a)\leq \sum_{j=2}^{n} A_j(X_a)+r(a)A_1(X_a).$$

By Wirtinger inequality we have $\sum_{j=1}^{n} A_j(Y_a)\geq c_{n-1}$ hence by the generalized Malliavin condition (2.9) and the Blaschke condition (2.10), we get by adding on a :

$$c_{n-1}\delta^{2n-2}\sum_{a\in S} r(a)^n\leq M+B,$$

and the corollary with $C=M+B$. □

3. Carleson measure and p-regularity.

3.1. **Geometric Carleson measures and p-regular domains.**

Let Ω be a domain in $\mathbb{C}^n$ with a good family of polydiscs $\mathcal{P}=\{P_a(\delta)\}$. Let $S(z,\zeta)$ be the Szegö kernel of Ω, i.e. the kernel of the orthogonal projection from $L^2(\partial\Omega)$ to $H^2(\Omega)$. It reproduces the functions in $H^2(\Omega)$:

$$\forall f\in H^{2}(\Omega),\ \forall a\in\Omega,\ f(a)=\langle f,\ k_{a}\rangle=\int_{\partial\Omega}f(\zeta)S(a,\zeta)\,d\sigma(\zeta),$$

with the alternative notation $k_a(\zeta):=\overline{S}(a,\zeta)$, which allows us to see k_a as the reproducing kernel for a in $H^2(\Omega)$.

Let σ be the Lebesgue measure on $\partial\Omega$; we already noticed that the polydisc $P_a(2)$ overflows the domain Ω.

We already know what is a geometric Carleson measure by the definition 1.7 and a p-regular domain by the definition 1.6.

Remark 3.1. *Because $\left\Vert k_a\right\Vert_{p'}$ is a growing function of p', we have that if $(\Omega,\ \mathcal{P})$ is p-regular, then it is q-regular for $q>p$. Hence the 1-regularity implies the p-regularity for all $p\geq 1$.*

Remark 3.2. *If Ω is stricly pseudo-convex, then it is 1-regular. We shall see, in the next section, that the convex domains of finite type of $\mathbb{C}^n$ equipped with the family $\mathcal{P}$ of McNeal polydiscs are also 1-regular.*

The definition of dual boundedness is also known by the definition 1.8, so now we shall recall and prove theorem 1.9.

Theorem 3.3. *Let Ω be a strongly pseudo-convex domain in $\mathbb{C}^n$ with a good family $\mathcal{P}$ of polydiscs and which is p-regular with respect to $\mathcal{P}$. If the sequence of points $S\subset\Omega$ is dual bounded in $H^p(\Omega)$, then the measure $\nu:=\displaystyle\sum_{a\in S}r(a)^{1+2\mu(a)}\delta_a$, with $\mu(a):=\sum_{j=2}^{n}\frac{1}{m_j(a)}$, is a geometric Carleson measure in Ω.*

Proof.
Dual boundedness means that we have a sequence $\{\rho_a\}_{a\in S}\subset H^p(\Omega)$ such that

$$\rho_a(b)=\delta_{ab}\left\Vert k_a\right\Vert_{p'},\ \forall a\in S,\ \left\Vert\rho_a\right\Vert_p\leq C.$$

This implies that

$$\left\Vert\rho_a/\rho_a(a)\right\Vert_{H^p(\Omega_a)}\leq C\left\Vert k_a\right\Vert_{p'}^{-1}. \tag{3.1}$$

Then, as in [1], for a fixed $a\in\Omega$, we build a pseudo-convex smoothly bounded domain Ω_a limited by $P_a(2)\cap\partial\Omega$. The function $\rho_a/\rho_a(a)$ is in $H^p(\Omega_a)$ which is contained in the Nevanlinna class $\mathcal{N}(\Omega_a)$, it is zero on $S_a:=(S\backslash\{a\})\cap\Omega_a$ and is 1 at the point a, which is the right normalization by (2.6). Its $H^p(\Omega_a)$ norm is bounded by $\left\Vert k_a\right\Vert_{p'}^{-1}$ by (3.1). The p-regularity of Ω gives $\left\Vert k_a\right\Vert_{p'}^{-1}\lesssim\sigma(Q_a)$. Using the theorem 1.5 for the domain Ω_a, because the sequence S_a is contained in the zero set X which is in the Blaschke class, the measure associated to the sequence S_a is bounded by $\left\Vert\rho_a/\rho_a(a)\right\Vert_{\mathcal{N}(\Omega_a)}$. So we have the chain of inequalities

$$\sum_{b\in S\cap\Omega_a}r(b)^{1+2\mu(b)}\lesssim\left\Vert\rho_a/\rho_a(a)\right\Vert_{\mathcal{N}(\Omega_a)}\lesssim\left\Vert\rho_a/\rho_a(a)\right\Vert_{H^p(\Omega_a)}\lesssim\left\Vert k_a\right\Vert_{p'}^{-1}\lesssim\sigma(Q_a),$$

and the theorem. $\square$

3.2. The 1-regularity for convex domains of finite type.

Let r be a defining function for Ω, $\Omega:=\{z\in\mathbb{C}^n\ s.t.\ r(z)<0\}$, where Ω is a convex domain of finite type.

We shall use the notations of McNeal and Stein [22] :

$\forall x,y\in\partial\Omega,\ \rho(x,y)$ is the pseudo-distance, introduced by McNeal [19], which gives a structure of space of homogeneous type to $\partial\Omega$.

The "distance" in $\Omega,\ \rho^{*}(z,w)$ is defined by:

$$\rho^{*}(z,w):=\left|r(z)\right|+\left|r(w)\right|+\rho(\pi(z),\ \pi(w)),$$

where π is the normal projection on the boundary $\partial\Omega$ of Ω and ρ is the pseudo-distance on $\partial\Omega$.

Also McNeal introduced a family of polydiscs associated to the pseudo-distance and they defined a "tent" as the intersection of a polydisc of the family with $\overline{\Omega}$. These "tents" are what we called "Carleson windows" used in the previous section.

$T(z,w)$ is the smallest "tent" containing the points z and w.
The ball associated to the pseudo-distance ρ of center $x\in\partial\Omega$ and radius δ will be denoted $B(x,\delta)$. We note $d\sigma$ the Lebesgue measure on $\partial\Omega$ and dm the Lebesgue measure on $\mathbb{C}^{n}$.

Finally let $S(z,w)$ the Szegö kernel of Ω, i.e. the kernel associated to the orthogonal projection from $L^{2}(\partial\Omega)$ onto $H^{2}(\Omega)$.

We have([22], p 521) :

$$\forall(x,y)\in\partial\Omega\times\partial\Omega\backslash\Delta,\ \left|S(x,y)\right|\lesssim\frac{1}{\sigma(B(x,\delta))},\ \delta=\rho(x,y).$$

And :

$$\forall(z,w)\in\Omega\times\Omega\backslash\Delta,\ \left|S(z,w)\right|\lesssim\frac{\delta}{m(T(z,w))},\ \delta:=\rho^{*}(z,w).$$

Theorem 3.4. *If Ω is a convex domain of finite type in $\mathbb{C}^{n}$, then Ω is 1-regular.*

Proof.
Let $z\in\Omega,\ x=\pi(z)\in\partial\Omega$ be fixed and cover $\partial\Omega$ by annulus $C_{k}:=B(x,\ N^{k+1}\delta)\backslash B(x,\ N^{k}\delta),\ k\in\mathbb{N}$ union $B(x,\ \delta)$ with $\delta=\rho^{*}(z,\ z)=2\left|r(z)\right|$. We have the following estimates from ([22], p 525)

$$\sigma(B(x,\ \delta))\simeq\delta\prod_{j=2}^{n}\tau_{j}(x,\ \delta)^{2}\ ;\ m(T(z,\ \delta))\simeq\delta^{2}\prod_{j=2}^{n}\tau_{j}(x,\ \delta)^{2}\simeq\delta\sigma(B(x,\delta)).$$

Using this, we have

$$\sigma(B(x,\delta))\lesssim\frac{1}{N^{2-1}}\sigma(B(x,\ N\delta)),$$

hence, choosing N big enough to absorb the constant in the $\lesssim$ we get

$$\sigma(B(x,\delta))\leq\frac{1}{2}\sigma(B(x,\ N\delta))\Longrightarrow\sigma(C_{k})\geq\frac{1}{2}\sigma(B(x,\ N^{k+1}\delta)\geq\frac{1}{2}2^{k+1}\sigma(B(x,\delta)).\tag{3.2}$$

We shall use this to get estimate of the $L^{p}(\sigma)$ norm of the Szegö kernel:

$$\left\|S(z,\cdot)\right\|_{p}^{p}=\int_{\partial\Omega}\left|S(z,\ y)\right|^{p}\,d\sigma(y)\leq\frac{\delta^{p}}{m(T(z,x))^{p}}\sigma(B(z,\delta))+\sum_{k\in\mathbb{N}}\frac{N^{p(k+1)}\delta^{p}}{m(T(z,\ N^{k+1}\delta)^{p}}\sigma(C_{k}).$$

From $m(T(z,\ \delta))\simeq\delta\sigma(B(x,\delta))$, we get

$$m(T(z,\ N^{k+1}\delta)^{p}\simeq N^{p(k+1)}\delta^{p}\sigma(B(x,\ N^{k+1}\delta)),$$

hence

$$\left\|S(z,\cdot)\right\|_{p}^{p}\lesssim\sigma(B(x,\delta))^{1-p}+\sum_{k\in\mathbb{N}}\frac{\sigma(C_{k})}{\sigma(B(x,\ N^{k+1}\delta)^{p}}\lesssim\sigma(B(x,\delta))^{1-p}+\sum_{k\in\mathbb{N}}\sigma(C_{k})^{1-p},$$

because $\sigma(C_{k})\leq\sigma(B(x,\ N^{k+1}\delta))$.
Now, using (3.2), we get

$$\sigma(C_{k})\geq 2^{k}\sigma(B(x,\delta)),$$

hence, for $p>1$,

$$\sigma(B(x,\delta))^{1-p}+\sum_{k\in\mathbb{N}}\sigma(C_k)^{1-p}\lesssim$$
$$\lesssim \sigma(B(x,\delta))^{1-p}+\sum_{k\in\mathbb{N}}\tfrac{1}{2^{k(p-1)}}\sigma(B(x,\ \delta))^{1-p}\leq C_p^p\sigma(B(x,\ \delta))^{1-p},$$

with $C_p^p:=1+\sum_{k\in\mathbb{N}}2^{(1-p)k}=1+\dfrac{1}{1-2^{1-p}}.$

So we have the estimate:

$$\left\|S(z,\cdot)\right\|_p^p\lesssim C_p^p\sigma(B(x,\delta))^{1-p}\Longrightarrow\left\|S(z,\cdot)\right\|_p\lesssim C_p\frac{1}{\sigma(B(x,\ \delta))^{1/p'}},$$

where p' is the conjugate exponent of p.

So we get the theorem because $C_p\longrightarrow 2$ when $p\longrightarrow\infty$, hence $\sigma(B(z,\ \delta))\lesssim\dfrac{1}{\left\|S(z,\cdot)\right\|_\infty}$. □

3.3. **Structural hypotheses for convex domains of finite type.**

In the previous section we have shown :

$$\left\|k_a\right\|_p\lesssim\frac{1}{\sigma(B(a,\delta))^{1/p'}}.. \tag{3.3}$$

In fact we shall get the more precise result :

Theorem 3.5. *We have* $\left\|k_a\right\|_{H^p}\simeq\dfrac{1}{\sigma(B(\alpha,\ \delta))^{1/p'}}.$

Proof.

We have

$$\left\|k_a\right\|_{H^p}=\sup\{|f(a)|=|\langle f,\ k_a\rangle|\ \ s.t.\ f\in H^{p'}(\Omega),\ \left\|f\right\|_{p'}=1\},$$

hence to have the inequality in the other direction, we have to choose a good test function. The idea is to take the Bergman kernel of Ω, with its estimates given by McNeal in [20]. So let $K_\Omega(z,w)$ be the Bergman kernel of Ω, we have a lower bound([20], theorem 3.4):

$$K_\Omega(a,a)\gtrsim\prod_{j=1}^n\tau_j(a,\ \delta)^{-2}\simeq\frac{1}{\delta\sigma(B(\alpha,\delta))}, \tag{3.4}$$

here with $\delta=|r(a)|$ and a in a neighbourhood $\mathcal{U}_p$ of the point $p\in\partial\Omega$ and $\alpha=\pi(a)$. We also have an upper bound([20], theorem 5.2):

$$K_\Omega(a,z)\lesssim\prod_{j=1}^n\tau_j(a,\ \delta)^{-2}\simeq\frac{1}{m(T(z,a))},$$

always in a neighbourhood of uniform size of $p\in\partial\Omega$, and here with

$$\delta=|r(a)|+|r(z)|+M(a,z)\simeq\rho^*(a,z).$$

So, with $\alpha\in\partial\Omega$ fixed, $\pi(a)=\alpha$, and $\mathcal{U}$ a neighbourhood of α valid for these two estimates, we have

Lemma 3.6. $\displaystyle\int_{\mathcal{U}\cap\{r(z)=-\epsilon\}}|K_\Omega(a,z)|^p\ d\sigma(z)\lesssim\frac{C}{\delta^p\sigma(B(\alpha,\ \delta))^{p-1}}+c.$

Proof of the lemma.

We do it with the annulus $C_k:=B(x,\ N^{k+1}\delta)\backslash B(x,\ N^k\delta),\ k\in\mathbb{N}$ we already use in the proof of theorem 3.4

$$\int_{\mathcal{U}\cap\{r(z)=-\epsilon\}} |K_\Omega(a,z)|^p \, d\sigma(z) \leq$$
$$\leq \frac{1}{\delta^p\sigma(B(\alpha,\delta))^p}\sigma(B(z,\delta)) + \sum_{k\in\mathbb{N}} \frac{N^{p(k+1)}}{m(T(z,\ N^{k+1}\delta)^p}\sigma(C_k).$$

But we have

$$m(T(z,x)) \simeq \delta\sigma(B(x,\delta)) \text{ and } \sigma(C_k) \simeq \frac{1}{2^n\delta}m(T(z,\ x_n)).$$

From $m(T(z,\ \delta)) \simeq \delta\sigma(B(x,\delta))$, we get

$$m(T(z,\ N^{k+1}\delta)^p \simeq N^{p(k+1)}\delta^p\sigma(B(x,\ N^{k+1}\delta)),$$

hence

$$\int_{\mathcal{U}\cap\{r(z)=\epsilon\}} |K_\Omega(a,z)|^p \, d\sigma(z) \leq \frac{1}{\delta^p\sigma(B(\alpha,\delta))^{p-1}} + \sum_{n\in\mathbb{N}} \sigma(C_k)^{1-p}.$$

The same way as in the proof of theorem 3.4, we get

$$\int_{\mathcal{U}\cap\{r(z)=\epsilon\}} |K_\Omega(a,z)|^p \, d\sigma(z) \lesssim C\frac{1}{\delta^p\sigma(B(x,\ \delta))^{p-1}}.$$

Ouside of $\mathcal{U},\ K_\Omega(a,\ \cdot)$ is bounded because by [21], p 178 :

$$|K_\Omega(a,\ z)| \lesssim \frac{1}{m(T(a,z))},$$

and if $z\notin\mathcal{U}$ then $1 \lesssim m(T(a,z))$ uniformly in $a\in\Omega$.
Hence :

$$\|K_\Omega(a,\cdot)\|_{H^p}^p = \int_{\mathcal{U}\cap\{r(z)=\epsilon\}} |K_\Omega(a,z)|^p \, d\sigma(z) + \int_{(\partial\Omega\setminus\mathcal{U})\cap\{r(z)=\epsilon\}} |K_\Omega(a,z)|^p \, d\sigma(z) \leq$$
$$\leq \frac{C}{\delta^p\sigma(B(\alpha,\ \delta))^{p-1}} + c,$$

and the lemma. □

End of the proof of theorem 3.5.

Using the lower bound (3.4) and the previous lemma, we get

$$\frac{K_\Omega(a,a)}{\|K_\Omega(a,\cdot)\|_{H^{p'}}} \geq \frac{1}{\delta\sigma(B(\alpha,\delta))} \times \delta\sigma(B(\alpha,\ \delta))^{1/p} \geq \frac{1}{\sigma(B(\alpha,\ \delta))^{1/p'}}.$$

Hence

$$\|k_a\|_{H^p} = \sup\{|f(a)| = |\langle f,\ k_a\rangle| \ \ s.t.\ f\in H^{p'}(\Omega),\ \|f\|_{p'}=1\} \geq \frac{1}{\sigma(B(\alpha,\ \delta))^{1/p'}},$$

by the choice of $f(z) := \dfrac{K_\Omega(a,z)}{\|K_\Omega(a,\cdot)\|_{H^{p'}}}$.

Together with the inequality (3.3) we have proved the theorem. □

We get easily the structural hypotheses for the domain Ω.

Corollary 3.7. *If Ω is a convex domain of finite type in $\mathbb{C}^n$, then the structural hypotheses $SH(q)$ and $SH(p,s)$ are true, i.e. $\forall q\in]1,\infty[$,*

$$SH(q): \qquad \|k_a\|_q\|k_a\|_{q'} \lesssim \|k_a\|_2^2,$$

and, for $\forall p,s\in[1,\infty]$, $\dfrac{1}{s}=\dfrac{1}{p}+\dfrac{1}{q}$,

$$SH(p,s): \qquad \|k_a\|_{s'} \lesssim \|k_a\|_{p'}\|k_a\|_{q'}.$$

Proof.
The previous theorem gives

$$\|k_a\|_p \simeq \frac{1}{\sigma(B(\alpha,\ \delta))^{1/p'}}.$$

hence, just replacing,

$$\|k_a\|_q \|k_a\|_{q'} \simeq \|k_a\|_2^2,\ \|k_a\|_{s'} \simeq \|k_a\|_{p'} \|k_a\|_{q'}$$

and the corollary. $\square$

If the balls $B(\alpha,\ r) := P_a(2) \cap \partial\Omega$, with $r = r(a)$ and $\pi(a) = \alpha$, are pseudo-balls which give to $\partial\Omega$ a structure of space of homogeneous type, then we know [3] that if μ is a geometric Carleson measure, we have a Carleson embedding : $\forall p \in [1,\ \infty[,\ H^p(\Omega) \subset L^p(\mu)$.

But we know that a convex domain of finite type in $\mathbb{C}^n$ is equipped with a structure of space of homogeneous type by [20], hence a geometric Carleson measure μ gives an embedding of $H^p(\Omega)$ into $L^p(\mu)$, hence if S is dual bounded in any $H^p(\Omega)$ for a $p \geq 1$, then S is a Carleson sequence. If a sequence of points $S \subset \Omega$ is dual bounded in $H^p(\Omega)$, then the associated measure μ_S is a geometric Carleson measure by the 1-regularity of Ω, hence S is a Carleson sequence. Morever the structural hypotheses are true for Ω, a convex domain of finite type in $\mathbb{C}^n$, by the previous corollary, hence we are in position to apply theorem 6.1 and theorem 6.2 of [2]:

Theorem 3.8. *If Ω is a convex domain of finite type in $\mathbb{C}^n$ and if $S \subset \Omega$ is a dual bounded sequence of points in $H^p(\sigma)$, then, for any $q < p,\ S$ is $H^q(\Omega)$ interpolating with the linear extension property, provided that $p = \infty$ or $p \leq 2$.*

4. Examples of strongly pseudo-convex domains.

The examples are mainly based on the theorem 1.12. We shall recall and prove it here.

Let Ω be a domain in $\mathbb{C}^n$ and $\mathcal{L}$ its Lévi form. Set $\mathcal{D} := \det\mathcal{L}$; then the set W of points of weak pseudo-convexity is $W := \{z \in \partial\Omega\ s.t.\ \mathcal{D}(z) = 0\}$.

Theorem 4.1. *Let Ω be a domain in $\mathbb{C}^n$ of finite linear type, and $\mathcal{D}$ the determinant of its Lévi form. Suppose that:*

$$\forall \alpha \in \partial\Omega,\ \exists v \in T_\alpha^{\mathbb{C}}(\partial\Omega)\ s.t.\ \exists k \in \mathbb{N},\ \frac{\partial^k \mathcal{D}}{\partial v^k}(\alpha) \neq 0,$$

then Ω is strongly pseudo-convex and can be equipped with a family of polydiscs whose multi-type is the given linear multi-type.

Proof.

The fact that there is a good family of polydiscs associated to the linear type is given by the theorem (2.3).

It remains to verify the condition on the smallness of the set W on complex lines tangent to $\partial\Omega$ to have that Ω is strongly pseudo-convex.

Let $\alpha \in \partial\Omega$, we may suppose that $\alpha = 0$ and that the complex normal is the z_1 axis. The complex tangent lines are those of $z_j = x_j + y_j,\ 1 < j \leq n$.

Because Ω fullfills the hypothesis of the theorem, there is a $1 < j \leq n$, a real direction, for instance y_j, in the complex line z_j, and an integer m, such that, with $\tilde{\mathcal{D}}$ being the restriction of $\mathcal{D}$ to the line z_j via the diffeomorphism $\pi,\ \tilde{\mathcal{D}} := \mathcal{D} \circ \pi$:

$$\frac{\partial^m \tilde{\mathcal{D}}}{\partial y_j^m}(0) = \frac{\partial^m \mathcal{D}}{\partial y_j^m}(0) \neq 0.$$

The differentiable preparation theorem of Malgrange gives:

there is a polynomial with $\mathcal{C}^\infty$ coefficients,

$P(x_j,\ y_j) = y_j^m + \sum_{k=1}^m a_k(x_j) y_j^{m_j-k}$

and a $\mathcal{C}^\infty$ function $Q(x_j,\ y_j)$ such that $Q(0)\neq 0$ and

$\tilde{\mathcal{D}}(x_j,\ y_j) = Q(x_j,\ y_j)P(x_j,\ y_j)$.

Hence the zero set of $\tilde{\mathcal{D}}$ is the same as the one of P and we know by corollary 5.4 that the upper Minkowski dimension of it is less or egal to $2 - \dfrac{1}{m}$.

Because $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are $\mathcal{C}^\infty$ functions, $\dfrac{\partial^m \tilde{\mathcal{D}}}{\partial y_j^m}\neq 0$ in a neighbourhood of 0 with the same number m, hence we have that the upper Minkowski dimension of $\{\tilde{\mathcal{D}} = 0\}$ is less or egal to $2 - \dfrac{1}{m}$ in all the slices parallel to the z_j axis in a neighbourhood of 0 and we are done. $\square$

4.1. **Convex domains.**

Theorem 4.2. *Let Ω be convex in a neighbourhood of $0 \in \mathbb{R}^{n+1}$. Suppose that the tangent space at 0 is $x_{n+1} = 0$ and $\partial\Omega = \{x_{n+1} = f(x_1, ..., x_n)\}$, with f convex. If the determinant of the hessian of f is flat at 0 then f is flat in a direction $x = (x_1,\ ...,\ x_n) \in \mathbb{R}^n$ of the tangent space at 0.*

Proof.

If f is not flat in any direction, we can find $\alpha > 0$ and $m \in \mathbb{N}$ such that $f(x) \geq \alpha\,|x|^{2m}$ in a ball $B(0,\ R) \subset \mathbb{R}^n$. Let us see the functions

$h(x) := \dfrac{\alpha}{2}\,|x|^{2m},\ g(x) := \dfrac{\alpha}{2}\,|x|^{2m} + \epsilon\,|x|^2 + \delta,$

with $\epsilon > 0$ and $\delta > 0$. Denote H_f the hessian of the function f.

Because $\det H_f$ is flat at 0, there is a ball $B(0,\ r) \subset \mathbb{R}^n$ such that:

$\forall x \in B(0, r),\ \det H_f(x) \leq \det H_h(x)$

and

$$\forall \epsilon > 0,\ \forall \delta > 0,\ \det H_h < \det H_g. \tag{4.1}$$

We choose ϵ and δ so small that there is a real t such that

$r > t > \sqrt[1/2m]{2\dfrac{\delta + \epsilon r^2}{\alpha}},$

then

$\dfrac{\alpha}{2}t^{2m} > \epsilon t^2 + \delta \Longrightarrow \alpha t^{2m} > \dfrac{\alpha}{2}t^{2m} + \epsilon t^2 + \delta,$

hence,

$$\forall x \ s.t.\ |x| = t,\ f(x) \geq \alpha\,|x|^{2m} > g(x). \tag{4.2}$$

On the other hand, because $g(0) = \delta > f(0) = 0$, and f and g are continuous, we get

$\exists s > 0,\ s < t\ s.t.\ \forall x,\ |x| < s,\ f(x) < g(x).$

The maximum principle for the Monge-Ampère operator says [5].

Lemma 4.3. *Let a convex function v defined in a bounded open set V (i.e. $H_v \geq 0$) and a regular function ρ such that*

$$\det H_v(x) > \det H_\rho(x),\ v \leq \rho \ on\ \partial V,$$
then $v \leq \rho$ *on* V.

Because $\det H_g > \det H_f$ in $B(0,r)$ by (4.1) and $g < f$ on $\partial B(0,t)$ by (4.2), we can apply this principle, i.e. $g \leq f$ everywhere in $B(0,\ t)$ which is a contradiction in the ball $B(0,s)$. Hence f has to be flat in some direction. □

Corollary 4.4. *Let* Ω *be a convex domain in a neighbourhood of* $0 \in \partial\Omega \subset \mathbb{R}^n$. *If* $\partial\Omega$ *is flat in no direction of it tangent space at* 0, *then the determinant of the hessian of* Ω *is not flat at* 0.

Proof.
If not we have a contraction with theorem 4.2. □

Let us see now the case of a convex domain of finite type in $\mathbb{C}^n$.

Corollary 4.5. *Let* Ω *be a convex domain of finite type in* $\mathbb{C}^n$ *near* $0 \in \partial\Omega \subset \mathbb{C}^n$. *There is a complex line* L *in the tangent complex space at* 0 *and a real vector* $v \in \mathbb{C}L$, *such that the determinant of the Lévi form of a defining function for* Ω *near* 0 *is non flat in the direction* v.

Proof.
Let $L_1, ...,\ L_{n-1}$ be an orthonormal basis of $T_0^{\mathbb{C}}\partial\Omega$ (here the complex normal direction is L_n). Because Ω is of finite type, we know that in any complex direction $L_j,\ j < n$, there is at most one real direction w_j in which $\partial\Omega$ is flat. If such a direction does not exist we take any real direction w_j in L_j. We then cut the convex Ω by the subspace $E := \{w_1 = \cdots = w_{n-1} = 0\}$.

We notice that we can always suppose that, in $L_j,\ z_j = x_j + iy_j,\ j = 1, ..., n-1$, the eventually flat directions w_j are the y_j, without changing the ambiant complex structure. Let
$$\tilde{\rho}(x) := x_n - \tilde{f}(x) := \rho(x,\ 0) = x_n - f(x,\ 0),$$
with, of course $f(x,\ 0) = f(x_1,\ ...,\ x_{n-1})$, i.e. we set $y = (y_1,\ ...,\ y_{n-1}) = 0$.
Then

$$\partial\overline{\partial} f(x,0) = \mathcal{L}(x,\ 0) = \{\frac{\partial^2 f}{\partial x_j \partial x_k}(x,\ 0)\}_{j,k=1,...,\ n-1} = H_{\tilde{f}}(x) \tag{4.3}$$

and the new convex set $\Omega_1 := \Omega \cap E$ still verifies the conditions of corollary 4.4 : $\tilde{\mathcal{D}}(x) := \det H_{\tilde{f}}(x)$ is not flat, hence there is a real vector v in the tangent space at 0 for $\partial\Omega_1$ such that $\tilde{\mathcal{D}}$ is not flat in the direction v. This means
$$\exists k \in \mathbb{N}\ s.t.\ \frac{\partial^k \tilde{\mathcal{D}}}{\partial v^k}(0) \neq 0\ ;$$
but, using (4.3), we get
$$\frac{\partial^k \mathcal{D}}{\partial v^k}(0) = \frac{\partial^k \tilde{\mathcal{D}}}{\partial v^k}(0) \neq 0,$$
and the corollary. □

Theorem 4.6. *Let* Ω *be a convex domain of finite type in* $\mathbb{C}^n$; *then* Ω *is strongly pseudo-convex.*

Proof.
By use of corollary 4.5, it remains to apply theorem 1.12. □

4.2. **Locally diagonalizable domains.**

In this context, the domains with a locally diagonalizable Lévi-form where introduced by C. Fefferman, J. Kohn and M. Machedon [9] in order to obtain Hölder estimates for the $\overline{\partial}_b$ operator.

We shall need the following lemma.

Lemma 4.7. *Let Ω be a domain locally diagonalizable in $\mathbb{C}^n$ and of finite linear type. Then the determinant of its Lévi form is not flat on the complex tangent space of $\partial\Omega$.*

Proof.
Let $\alpha \in \partial\Omega$, then the fact that Ω is locally diagonalizable means that there is a neighbourhood V_α of α and $(L_1, ..., L_n)$ a basis of $\mathbb{C}^n$ depending smoothly on $z \in V_\alpha$, and diagonalizing the Lévi form $\mathcal{L}$, with L_1 the complex normal direction, so we have, restricting $\mathcal{L}$ to the complex tangent space:

$$\mathcal{L}(z) = \begin{pmatrix} \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Hence $\mathcal{D} := \det\mathcal{L} = \lambda_2 \cdots \lambda_n$. Now suppose that, for any complex direction $L_j,\ j = 2, ...,\ n$, at α, there is a real direction $v_j,\ v_j \in L_j$, such that $\exists k = k_j \in \mathbb{N},\ \dfrac{\partial^k \lambda_j}{\partial v_j^k}(\alpha) \neq 0$, then with $k := (k_2, ...,\ k_n)$:

$$\frac{\partial^{|k|}\mathcal{D}}{\partial^{k_2} v_2 \cdots \partial^{k_n} v_n}(\alpha) \neq 0,$$

and $\mathcal{D}$ is not flat at α. Hence if $\mathcal{D}$ is flat at α, we must have

$$\exists L_j,\ \forall v_j \in L_j,\ \forall k \in \mathbb{N},\ \frac{\partial^k \lambda_j}{\partial v_j^k}(\alpha) = 0.$$

But $\dfrac{\partial^k \lambda_j}{\partial v_j^k}(\alpha) = \dfrac{\partial^{k-2}\mathcal{L}_{jj}}{\partial v_j^{k-2}}(\alpha) = 0$ and this is a contradiction with the fact that Ω is of finite linear type. □

Now we have

Theorem 4.8. *Let Ω be a domain locally diagonalizable in $\mathbb{C}^n$ and of finite linear type. Then Ω is strongly pseudo-convex.*

Proof.
As for the convex domains, by use of the lemma 4.7, it remains to apply theorem 1.12. □

4.3. **Domains with real analytic boundary.**

Lemma 4.9. *Let Ω be a bounded domain with real analytic boundary, then Ω is of finite linear type.*

Proof.
Take a point $\alpha \in \partial\Omega$ and suppose that a real line through α has a contact of infinite order with $\partial\Omega$, then, using Lojasiewicz [27] we get that the line, which is real analytic, and $\partial\Omega$ are regularly situated, hence the line must be contained in $\partial\Omega$. But this cannot happen because $\partial\Omega$ is bounded. □

In fact we have a better result because we know, by the work of K. Diederich and J-E. Fornaess [13], that Ω is of finite type.

The function $\mathcal{D} = \det\mathcal{L}$ is also real analytic, hence if $\mathcal{D}$ is flat at a point $\alpha \in \partial\Omega$, this means in particular that $\forall v \in T_\alpha(\partial\Omega),\ \forall k \in \mathbb{N},\ \dfrac{\partial^k \mathcal{D}}{\partial v^k}(\alpha) = 0$, hence $\mathcal{D}$ is identically zero on $\partial\Omega$. This says

that all the points of $\partial\Omega$ are non stricly pseudo-convex points. But this is impossible because $\partial\Omega$ is compact, hence contains at least a strictly pseudo-convex point, because of the following simple and certainly well known lemma:

Lemma 4.10. *Let Ω be a bounded domain in $\mathbb{R}^n$, with a smooth boundary of class $\mathcal{C}^3$. Then $\partial\Omega$ contains a point of strict convexity.*

Proof.
Let $a \in \Omega$ be fixed and let $\alpha \in \partial\Omega$ be such that $\left|\alpha - a\right|^2 = \sum_{j=1}^{n} (\alpha_j - a_j)^2$ is maximal ; such a point exists because $\partial\Omega$ is compact. Then, near α, $\partial\Omega$ lies inside the ball of center a and radius $\left|\alpha - a\right|$. Now by rotation and translation we can suppose that $a = 0$ and near a we have

$$x \in \partial\Omega \Longleftrightarrow x_1 = f(x'),\ x' = (x_2, ..., \ x_n),\ f(0) = f'(0) = 0.$$

By Taylor we get

$$f(x') = \frac{1}{2} \langle H_f(0)x',\ x'\rangle + o(|x'|^2),$$

and the hypothesis implies that, for x' near 0, $f(x') \geq \beta \left|x'\right|^2$, with $\beta > 0$. This implies that $H_f(0) \geq \beta I_d$ and 0 is a point of strict convexity for $\partial\Omega$. □

Now let $\alpha \in \partial\Omega$ and suppose that $\mathcal{D}$ is flat in all the complex tangent directions of $T_\alpha^{\mathbb{C}}(\partial\Omega)$, then, because $\partial\Omega$ is of finite type, we can recover the derivatives in the "missing direction", namely the real direction conjugate to the normal one, by brackets of derivatives in the complex tangent directions.
Hence we have that $\mathcal{D}$ is also flat in the direction conjugate to the normal one, but this will implies that $\mathcal{D}$ is flat at the point α, and this is forbidden by the lemma. So we can apply theorem 1.12 to conclude:

Theorem 4.11. *Let Ω be a domain in $\mathbb{C}^n$ with real analytic boundary, then Ω is strongly pseudo-convex and of finite linear type.*

5. Appendix

5.1. Minkowski dimension.

5.1.1. *Definitions and first properties.* There are several equivalent definitions for the Minkowski-Bouligand dimension we shall just give the one which is relevant for us.

Let W be a bounded set in $\mathbb{R}^n$, $\epsilon > 0$ and $\mathcal{R}_\epsilon(W)$ a covering of W with euclidean ball of radius ϵ. We define

$$N_\epsilon(W) := \inf_{\mathcal{R}_\epsilon} \#\mathcal{R}_\epsilon(W),$$

where $\#\mathcal{R}_\epsilon(W)$ is the number of elements in $\mathcal{R}_\epsilon(W)$.

Now we have

Definition 5.1. *For a non-empty bounded subset W of $\mathbb{R}^n$ the upper Minkowski-Bouligand dimension is*

$$\overline{\dim}_M W := \inf\{s \ s.t. \ \overline{\lim_{\epsilon \longrightarrow 0^+}} N_\epsilon(W)\epsilon^s = 0\}.$$

In fact we shall use this via the following lemma.

Lemma 5.2. *If the upper dimension of a bounded set W is s, then we have*

$$\forall t > s,\ \forall \epsilon > 0,\ \exists C > 0,\ \exists \mathcal{R}_\epsilon(W) \ s.t. \ \#\mathcal{R}_\epsilon(W) \leq C\epsilon^{-t}.$$

Proof.
Because $t > s$ the definition implies

$$\overline{\lim_{\epsilon\longrightarrow 0^+}} N_\epsilon(W)\epsilon^t = 0,$$

which means that

$$\forall\eta > 0,\ \exists\delta > 0,\ \forall\epsilon < \delta,\ \exists\mathcal{R}_\epsilon(W)\ s.t.\ \#\mathcal{R}_\epsilon(W)\epsilon^t < \eta,$$

hence, with $\eta = 1$,

if $\epsilon < \delta,\ \exists\mathcal{R}_\epsilon(W)\ s.t.\ \#\mathcal{R}_\epsilon(W)\epsilon^t < 1$;

if $\epsilon \geq \delta$, we always have

$$N_\epsilon(W) \leq V\epsilon^{-n} \Longrightarrow N_\epsilon(W) \leq (V\delta^{-n+t})\epsilon^{-t},$$

with V the volume of a ball containing the bounded set W.
Hence taking $C = \max(1,\ V\delta^{-n+t})$, the lemma is proved. □

Lemma 5.3. *Let f be a function lipschitz $\alpha > 0$ on the closed unit cube $C = [0,\ 1]^n$ of $\mathbb{R}^n$. Then the graph $G := \{(x, y)\ s.t.\ x \in C,\ y = f(x)\} \subset \mathbb{R}^{n+1}$ of f has its upper Minkowski dimension bounded by $n + 1 - \alpha$.*

Proof.
we have, by assumption,

$$\forall x \in C,\ x + h \in C,\ |f(x + h) - f(x)| \leq c\,\|h\|^\alpha.$$

let $\epsilon > 0$; the graph of f above the cube $B(x,\ \epsilon)$ is contained in

$$E_x := \{B(x,\ \epsilon) \times [f(x) - c\epsilon^\alpha,\ f(x) + c\epsilon^\alpha].$$

To cover this set we need $2c\dfrac{\epsilon^\alpha}{\epsilon} = 2c\epsilon^{\alpha-1}$ intervals of lenght ϵ to cover $[f(x) - c\epsilon^\alpha,\ f(x) + c\epsilon^\alpha]$ in $\mathbb{R}$ and ϵ^{-n} cubes of side ϵ to cover C in $\mathbb{R}^n$. Hence we can cover G with $N_\epsilon(G) = 2c\epsilon^{-n-1+\alpha}$ cubes of side ϵ in $\mathbb{R}^{n+1}$. This being true for all $\epsilon > 0$, we get the lemma.

The following result is a corollary of a theorem of Ostrowski [24].

Corollary 5.4. *Let $P(y)$ be a monic polynomial of degree d in the real variable y whose coefficients are $\mathcal{C}^\infty$ functions of $x \in \mathbb{R}^n$. Then the graph of the zero set of P has its upper Minkowski dimension strictly less than $n + 1 - \dfrac{1}{d}$ over any compact set.*

Proof.
by a theorem of Ostrowski [24] we have that locally the roots y of the equation

$$P(y) = y^d + a_1y^{d-1} + \cdots + a_d = 0,$$

are Lipschitz $\dfrac{1}{d}$ functions of the coefficients a_j. Composing with the $\mathcal{C}^\infty$ function

$$x \longrightarrow a(x) := \{a_j(x),\ j = 1, ..., d\},$$

we get that the roots y are still Lipschitz $\dfrac{1}{d}$ and we can apply the previous lemma. □

5.1.2. *Domains in $\mathbb{C}^n$.*

Let $\mathbb{D}^n$ be the unit polydisc in $\mathbb{C}^n$; we have the lemma:

Lemma 5.5. *Let $W \subset \mathbb{D}^n \subset \mathbb{C}^n$ and $\alpha > 0$ such that the upper Minkowski dimension of*

$$W \cap \{z_2 = a_2,\ ...,\ z_n = a_n\}$$

is $2 - \alpha$ for all $a' = (a_2, ...,\ a_n) \in \mathbb{D}^{n-1}$. Let $s > 0$ and Γ a net of balls of radius s in $\mathbb{D}^n$. Let $S \subset \Gamma \cap W$ and let P_a be a polydisc centered on $a \in S$, with sides varying with $a \in S$ in direction, and with radii $(r,\ l_2r...,\ l_nr)$. Let $l = \max_{j=2,...,n} l_j$. Suppose that the polydiscs P_a are disjoint, then, for any $\alpha' < \alpha$, $\exists C_{\alpha'}$ s.t. $\sum_{a\in S} \mathrm{Area}(P_a) \leq C_{\alpha'} l^{\alpha'} r^{\alpha'}$.

Proof.

First set $C(b,\ lr)$ to be a polydisc in $\mathbb{D}^n$, centered at b and of side $(lr, ...,\ lr)$. The polydisc P_a with a in $C(b,\ lr)$ is contained in the "double" polydisc $C(b,\ 2lr)$ and hence the volume of the union of all those polydiscs P_a is bounded by the volume of $C(b,\ 2lr)$. These polydiscs being disjoint we get

$$\sum_{a\in S\cap C(b,\ lr)} \mathrm{Area}(P_a) \leq \mathrm{Area}(C(b,\ 2lr)) = 4^n\pi^n l^{2n}r^{2n}.$$

Each polydisc verifies $\mathrm{Area}(P_a) = \pi^n l_2^2\cdots l_n^2 r^{2n}$, hence the number of points N_C of S in $C(b,\ lr)$ can be estimated

$$N_C \leq 4^n\pi^n l^{2n}r^{2n}/\pi^n l_2^2\cdots l_n^2 r^{2n} = 4^n\frac{l^{2n}}{l_2^2\cdots l_n^2}.$$

Let $b' = (b_2, ..., b_n)$ be fixed, then we can find a net of polydiscs $C((b_1,\ b'),\ lr)$ such that the number n_B of these polydiscs which meet S is, by the Minkowski assumption,

$$n_B \leq C(lr)^{\alpha'-2}.$$

Hence the number N_B of points of S in a slice of "depth" lr verifies

$$N_B \leq n_B \times N_C \leq C(lr)^{\alpha'-2}\times 4^n\frac{l^{2n}}{l_2^2\cdots l_n^2}.$$

The number of such slices, when b' varies, is bounded by $1/(lr)^{2(n-1)}$, hence the total number N of points in S can be estimated

$$N \leq \frac{N_B}{l^{2(n-1)}r^{2(n-1)}} \leq 4^nCl^{\alpha'}r^{\alpha'}\frac{1}{l_2^2\cdots l_n^2 r^{2n}}.$$

Hence the total area of the polydiscs P_a is

$$A := \sum_{a\in S}\mathrm{Area}(P_a) = N\times\pi^n l_2^2\cdots l_n^2 r^{2n} \leq 4^n\pi^n Cl^{\alpha'}r^{\alpha'},$$

and this lemma. $\square$

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